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NATURAL VIBRATIONS OF HYDROTURBINE CASCADES HAVING SMALL GEOMETRIC NONUNIFORMITY

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In the theory of turbomachinery elasticity, it is well known that the introduction of even a small nonuniformity in cascades can affect the stability of blade vibrations in a flow and increase considerably the dynamic stresses in individual blades with forced vibrations of the cascades. For gas turbomachines, for which the natural frequencies of blade vibrations depend only slightly on the ambient medium, this problem has been adequately studied. For hydraulic turbines, there have been few studies in this direction. In the present work, we examine the effect of small geometric nonuniformity of hydraulic-turbine cascades on the natural frequencies and modes of their vibrations in a quiescent liquid.

1. Formulation of the Problem. We study free vibrations of a hydraulic-turbine cascade of sufficiently thin blades. The cascade is located between two coaxial cylinders in a quiescent, ideal, incompressible liquid. In contrast to the model studied in [1], we shall assume that the cascade has small geometric nonuniformity because of inaccurate manufacture and assembly.

The geometric parameters of the cascade are assumed to be specified. We introduce a homogeneous cascade with blades of identical thickness $h(\mathbf{r}_{0m})$, where $\mathbf{r}_{0m}$ are the radius vectors of the centroidal surfaces $S_m^{(0)}$ of the blades. The geometric nonuniformity of the cascade considered is then given by the functions

$$\delta\mathbf{r}_m(\mathbf{r}_{0m}) = \mathbf{r}_m - \mathbf{r}_{0m}, \qquad \delta h_m(\mathbf{r}_{0m}) = h_m(\mathbf{r}_m) - h(\mathbf{r}_{0m}). \tag{1.1}$$

Here $\mathbf{r}_m$ are the radius vectors of the corresponding points on the centroidal surfaces S_m of the specified blades and h_m are their thicknesses. The smallness of the geometric nonuniformity is characterized by the parameter

$$\varepsilon_1 = \max\{|\,\delta\mathbf{r}_m\,|/b,\ |\,\delta h_m\,|/h\} \ll 1, \tag{1.2}$$

where b is the characteristic blade-section chord.

The geometric nonuniformity of the cascade leads to mistuning of the natural frequencies of blade vibrations, which almost always occurs under real operating conditions of turbomachines. As a rule,

$$\max\left\{\frac{\omega_m^{(1)} - \omega^{(1)}}{\omega^{(1)}}\right\} = \varepsilon_2 \ll 1, \tag{1.3}$$

where $\omega_m^{(1)}$ is the natural frequency of first-mode vibrations of the mth blade the and $\omega^{(1)}$ is the natural frequency of the corresponding uniform cascade. The introduction of the second small parameter ε_2 is due to the fact the order of smallness of the quantities in relations (1.2) and (1.3) can be different.

Assuming that the interaction between the blades is realized only via the liquid, we obtain a matrix system of equations that describe the free vibrations of the blade cascade considered:

$$(\mathbf{C}_m - \lambda\mathbf{M}_m)\mathbf{X}_m = \lambda\sum_{n=0}^{N-1}\mathbf{A}_{mn}\mathbf{X}_n \qquad (\lambda = \omega^2, \quad m = 0, 1, \ldots, N-1). \tag{1.4}$$

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Here $\mathbf{X}_m$ are the vectors of the approximation coefficients of the amplitude function of vibrations of the centroidal surface of the mth blade over a certain system of basis functions [1], $\mathbf{C}_m$ and $\mathbf{M}_m$ are the stiffness and mass matrices, $\mathbf{A}_{mn}$ are the matrices of the hydrodynamic influence coefficients, ω is the natural frequency of blade vibrations, and N is the number of blades. We write the matrices of this equation in the form

$$\mathbf{C}_m = \mathbf{C} + \delta\mathbf{C}_m, \quad \mathbf{M}_m = \mathbf{M} + \delta\mathbf{M}_m, \quad \mathbf{A}_{mn} = \mathbf{A}_{mn}^{(0)} + \delta\mathbf{A}_{mn}, \tag{1.5}$$

where $\mathbf{C}$, $\mathbf{M}$, and $\mathbf{A}_{mn}^{(0)}$ are the matrices of the corresponding uniform cascade. Note that the matrices $\mathbf{A}_{mn}^{(0)}$ have the cyclicity property [2], i.e.,

$$\mathbf{A}_{mn}^{(0)} = \mathbf{L}_s, \tag{1.6}$$

where

$$s = \begin{cases} n - m & \text{for } n \geqslant m, \\ N + (n - m) & \text{for } n < m. \end{cases}$$

Since the values of the elements of the matrices $\mathbf{C}_m$, $\mathbf{M}_m$, and $\mathbf{A}_{mn}$ are continuous functions of the geometric parameters of the cascade, the perturbed components of these matrices are represented, according to (1.2), as

$$\delta\mathbf{C}_m = \varepsilon_1\tilde{\mathbf{C}}_m, \quad \delta\mathbf{M}_m = \varepsilon_1\tilde{\mathbf{M}}_m, \quad \delta\mathbf{A}_{mn} = \varepsilon_1\tilde{\mathbf{A}}_{mn} \tag{1.7}$$

$$\big[\|\tilde{\mathbf{C}}_m\| = O(\|\mathbf{C}\|), \quad \|\tilde{\mathbf{M}}_m\| = O(\|\mathbf{M}\|), \quad \|\tilde{\mathbf{A}}_{mn}\| = O(\|\mathbf{A}_{mn}^{(0)}\|)\big].$$

Here and below, we use the spectral norms of the matrices, subjected to Euclidean norms of the corresponding vectors [3]. The problem is to solve the generalized eigenvalue problem for system (1.4).

2. Tentative Estimate of Perturbations of Eigenvalues. The presence of small parameters in the formulated problem gives grounds to use the perturbation method to solve the problem. However, the effectiveness of this method depends on the conditioning number of eigenvalues, which determines the order of magnitudes of the desired eigenvalues with respect to the magnitude of perturbations of the initial data. For the generalized eigenvalue problem, to which the formulated problem is reduced, the general estimate of the conditioning number [2] is too crude to draw a tentative conclusion on the possibility of using this method. Indeed, we write system (1.4) with allowance for (1.5) and (1.7) as one matrix equation

$$(\mathbf{B} - \lambda\mathbf{A})\mathbf{X} = -\varepsilon_1(\tilde{\mathbf{B}} - \lambda\tilde{\mathbf{A}})\mathbf{X}, \tag{2.1}$$

where $\mathbf{A}$, $\tilde{\mathbf{A}}$, $\mathbf{B}$, and $\tilde{\mathbf{B}}$ are the block, real, symmetric matrices of the form

$$\mathbf{A} = \operatorname{diag}\{\mathbf{D}_m\}_0^{N-1} + \{\mathbf{A}_{mn}^{(0)}\}_0^{N-1}, \qquad \mathbf{D}_m = \mathbf{M},$$

$$\tilde{\mathbf{A}} = \operatorname{diag}\{\tilde{\mathbf{D}}_m\}_0^{N-1} + \{\tilde{\mathbf{A}}_{mn}\}_0^{N-1}, \qquad \tilde{\mathbf{D}}_m = \tilde{\mathbf{M}}_m,$$

$$\mathbf{B} = \operatorname{diag}\{\mathbf{B}_m\}_0^{N-1}, \quad \mathbf{B}_m = \mathbf{C}, \quad \tilde{\mathbf{B}} = \operatorname{diag}\{\tilde{\mathbf{B}}_m\}_0^{N-1}, \quad \tilde{\mathbf{B}}_m = \tilde{\mathbf{C}}_m.$$

We introduce the eigenvalues $\lambda_0^{(s)}$ ($s = 1, 2, \ldots, N_1$, where N_1 is the order of the matrices $\mathbf{A}$ and $\mathbf{B}$), and the eigenvectors $\mathbf{X}_0^{(s)}$ of the problem of free vibrations of the corresponding uniform cascade when the right side of Eq. (2.1) is equal to zero. We assume that the values of $\lambda_0^{(s)}$ are numbered in increasing order. Note that $\lambda_0^{(s)}$ are real positive numbers. We introduce left eigenvectors $\mathbf{Y}_0^{(s)}$ and number them so that

$$\mathbf{Y}_0^{(s)} = (\mathbf{X}_0^{(s)})^t, \qquad \|\mathbf{X}_0^{(s)}\| = 1. \tag{2.2}$$

Then, we obtain the relations

$$\mathbf{Y}_0^{(s)}\mathbf{B}\mathbf{X}_0^{(s)} = c_0^{(s)}, \quad \mathbf{Y}_0^{(s)}\mathbf{A}\mathbf{X}_0^{(s)} = m_0^{(s)}, \quad \lambda_0^{(s)} = c_0^{(s)}/m_0^{(s)}, \tag{2.3}$$

where $c_0^{(s)}$ determines the potential energy of blade deformations whose form corresponds to $\mathbf{X}_0^{(s)}$, and $m_0^{(s)}$ determines the generalized mass of the cascade vibrating by the same form. Applying the mean-value theorem to the integral expressions of these quantities, we find that they relate to the number of the corresponding eigenvalue of blade vibrations as follows:

$$m_0^{(s)} = O(m_0^{(1)}), \qquad c_0^{(s)} = \lambda_0^{(s)} O(c_0^{(1)})/\lambda_0^{(1)} \quad (s = 1, 2, \dots, N_1). \tag{2.4}$$

The mechanical meaning of the above dependence for $c_0^{(s)}$ is that the potential energy of the blade elements is proportional to the square of the curvature of the surfaces, which determines the vibration forms, and increases with increase in vibration mode number.

In accord with [2] and taking into account (2.3), we write the general estimate of the perturbations of the eigenvalues in the form

$$|\lambda - \lambda_0^{(s)}| \leqslant \varepsilon_1 \|\mathbf{B}\| \, \|\mathbf{A}^{-1}\| = \varepsilon_1 \max c_0^{(s)} / \min m_0^{(s)}. \tag{2.5}$$

From (2.5) it does not follow *a priori* that the perturbation of the eigenvalue will be small, because, first, there is a probability of poor conditioning of matrix $\mathbf{A}$, and, second, the value of max $c_0^{(s)}$ is not limited. According to (2.4), with increase in the number of basis functions N_1, which approximate the blade deformation, the value of $\max c_0^{(s)}$ will also increase because in the limit it will tend to infinity.

We show that, for the set Λ_0 of the eigenvalues $\lambda_0^{(s)}$ that correspond to low vibration modes, a more exact estimate is obtained based on the assumption (1.3).

For this, we examine the eigenvalue $\lambda^{(r)}$ of system (2.1) that satisfies the condition

$$\lambda_0^{(s_0-1)} \leqslant \lambda^{(r)} \leqslant \lambda_0^{(s_0)}, \qquad \lambda_0^{(s_0)} \in \Lambda_0 \qquad (\lambda_0^{(0)} = 0). \tag{2.6}$$

We represent the corresponding eigenvector $\mathbf{X}^{(r)}$ as a linear combination of the vectors $\mathbf{X}_0^{(s)}$:

$$\mathbf{X}^{(r)} = \sum_{s=1}^{N_1} \alpha_s^{(r)} \mathbf{X}_0^{(s)}, \qquad \|\mathbf{X}^{(r)}\| = 1. \tag{2.7}$$

Substituting (2.7) into (2.1), multiplying the left side of (2.1) by the vector $\mathbf{Y}_0^{(s)}$, and taking into account (2.3), we obtain

$$\alpha_s^{(r)} \lambda_r^{(s)} = \varepsilon_1 \Big(\frac{\tilde{c}_0^{(s)}}{c_0^{(s)}} - \frac{\lambda^{(r)}}{\lambda_0^{(r)}} \frac{\tilde{m}_0^{(s)}}{m_0^{(s)}} \Big), \tag{2.8}$$

where $\lambda_r^{(s)} = (\lambda^{(r)} - \lambda_0^{(s)})/\lambda_0^{(s)}$, $\tilde{c}_0^{(s)} = \mathbf{Y}_0^{(s)} \tilde{\mathbf{B}} \mathbf{X}^{(r)}$, and $\tilde{m}_0^{(s)} = \mathbf{Y}_0^{(s)} \tilde{\mathbf{A}} \mathbf{X}^{(r)}$.

Since the quantities $\tilde{c}_0^{(s)}$ and $\tilde{m}_0^{(s)}$ are continuous functions of the geometric nonuniformity parameters of the cascade δh_m and $\delta \mathbf{r}_m$, they can be divided, in a linear approximation, into two components:

$$\tilde{c}_0^{(s)} = \tilde{c}_h^{(s)} + \tilde{c}_\alpha^{(s)}, \qquad \tilde{m}_0^{(s)} = \tilde{m}_h^{(s)} + \tilde{m}_\alpha^{(s)}.$$

Here $\tilde{c}_h^{(s)}$ and $\tilde{c}_\alpha^{(s)}$ are the perturbed components of the potential strain energy of the cascade, the first of which depends only on the nonuniformity parameter δh_m and the second depends on $\delta \mathbf{r}_m$, $\tilde{m}_h^{(s)}$ is the perturbed component of the generalized mass of the cascade, which depends on δh_m, and $\tilde{m}_\alpha^{(s)}$ is the perturbed component of the attached mass of the liquid, which occurs because of the nonuniformity of the cascade in the parameter $\delta \mathbf{r}_m$. Taking into account (2.8), from condition (1.3) we obtain

$$2\varepsilon_2 \geqslant \varepsilon_1 \Big| \frac{\tilde{c}_h^{(1)}}{c_0^{(1)}} - \frac{\lambda^{(1)}}{\lambda_0^{(1)}} \frac{\tilde{m}_h^{(1)}}{m_0^{(1)}} \Big|, \quad 2\varepsilon_2 \geqslant \varepsilon_1 \Big| \frac{\tilde{c}_\alpha^{(1)}}{c_0^{(1)}} - \frac{\lambda^{(1)}}{\lambda_0^{(1)}} \frac{\tilde{m}_\alpha^{(1)}}{m_0^{(1)}} \Big|. \tag{2.9}$$

From (1.2) with allowance for (1.7) we find $\tilde{m}_h^{(1)}/m_0^{(1)} \leqslant 1$. For the perturbed component of the attached mass subject to condition (1.2), the estimate $\tilde{m}_\alpha^{(1)}/m_0^{(1)} = O(1)$ is valid.

Taking into account the estimates for $\tilde{m}_h^{(1)}$ and $\tilde{m}_\alpha^{(1)}$ and using inequalities (2.9) we obtain

$$\tilde{m}_0^{(1)}/m_0^{(1)} = O(1), \qquad \tilde{c}_0^{(1)}/c_0^{(1)} = O(\varepsilon)/\varepsilon_1, \qquad \varepsilon = \varepsilon_1 + \varepsilon_2. \tag{2.10}$$

From the integral expressions for the corresponding exact values of $\tilde{m}_0^{(s)}$ and $\tilde{c}_0^{(s)}$ we obtain relations similar to (2.4):

$$|\tilde{m}_0^{(s)}| = O(|\tilde{m}_0^{(1)}|), \qquad |\tilde{c}_0^{(s)}| = \frac{c_0^{(s)}}{c_0^{(1)}} O(|\tilde{c}_0^{(1)}|). \tag{2.11}$$

Taking into account (2.10) and (2.11), from (2.8) for $s \leqslant s_0$ we have

$$a_s^{(r)} \lambda_r^{(s)} = O(\varepsilon). \tag{2.12}$$

Multiplying (2.1) from the left by the vector $\mathbf{Y}^{(r)} = (\mathbf{X}^{(r)})^t$, we obtain

$$\sum_{s=1}^{N_1} (\alpha_s^{(r)})^2 \tilde{c}_0^{(s)} \lambda_r^{(s)} = \varepsilon_1 (\tilde{c}^{(r)} - \lambda^{(r)} \tilde{m}^{(r)}), \tag{2.13}$$

where $\tilde{c}^{(r)} = \mathbf{Y}^{(r)} \tilde{\mathbf{B}} \mathbf{X}^{(r)}$ and $\tilde{m}^{(r)} = \mathbf{Y}^{(r)} \tilde{\mathbf{A}} \mathbf{X}^{(r)}$. By analogy with (2.4) and (2.11), the integral expressions of the corresponding exact values of $\tilde{c}^{(r)}$ and $\tilde{m}^{(r)}$ lead to

$$|\tilde{m}^{(r)}| = O(|\tilde{m}_0^{(1)}|), \qquad |\tilde{c}^{(r)}| = \frac{c_0^{(r)}}{c_0^{(1)}} O(|\tilde{c}_0^{(1)}|). \tag{2.14}$$

Dividing (2.13) by $c_0^{(r)}$ with allowance for (2.4), (2.6), (2.10), (2.11), and (2.14) we obtain

$$\sum_{s=s_0}^{N_1} (\alpha_s^{(r)})^2 \lambda_r^{(s)} < \sum_{s=1}^{s_0-1} (\alpha_s^{(r)})^2 \lambda_r^{(s)} + O(\varepsilon). \tag{2.15}$$

To estimate the sum on the right side of (2.15), we divide this sum by two components:

$$\Pi_1 = \sum_{s=1}^{s_1} (\alpha_s^{(r)})^2 \lambda_r^{(s)}, \qquad \Pi_2 = \sum_{s=s_1+1}^{s_0-1} (\alpha_s^{(r)})^2 \lambda_r^{(s)}.$$

The value of s_1 in these expressions is chosen from the condition $\lambda_r^{(s_1)} = a \gg \varepsilon$. Applying the Hölder inequality to Π_2, using (2.12), and taking into account (2.2) and (2.7) we obtain

$$\Pi_1 = \frac{s_1}{a} O(\varepsilon^2), \qquad \Pi_2 = \sqrt{s_2}\, O(\varepsilon). \tag{2.16}$$

Here s_2 is the number of elements of the set

$$\mathbf{R}_2 = \{\lambda_0^{(s)} : (\lambda_0^{(s_0)} - \lambda_0^{(s)})/\lambda_0^{(s)} \leqslant a\}_{s=s_1+1}^{s_0-1}. \tag{2.17}$$

We assume that s_2 is small. Then, using the condition

$$s_1 \leqslant a/\varepsilon \tag{2.18}$$

and taking into account (2.16), from (2.15) we obtain the estimate

$$\min_s |\lambda_r^{(s)}| = O(\varepsilon), \qquad r \leqslant s_1 + s_2. \tag{2.19}$$

Thus, for the set

$$\Lambda_0 = \{\lambda_0^{(s)}\}_{s=1}^{s_0-1}, \tag{2.20}$$

where $s_0 = s_1 + s_2$, and s_1 and s_2 are determined from conditions (2.17) and (2.18), we have the asymptotic estimate of perturbations of the eigenvalues (2.19).

3. Use of the Perturbation Method. The standard perturbation method of solution of the eigenvalue problem of linear algebra [3] is based on the representation of the desired eigenvalues and eigenvectors in the form of expansions in powers of the small parameter ε. In this case, the zero terms of the expansions are their corresponding values for the unperturbed matrices. In the case considered, this approach is insufficiently effective for two reasons. First, there is a high probability that the unperturbed components of the eigenvalues have similar values. This makes incorrect the standard expansion of the eigenvectors. Second, as a consequence of the solution of the generalized rather than standard eigenvalue problem, the order of magnitude of perturbations of the eigenvectors differs from ε.

According to [4], we first find a comparison function for the asymptotic representation of the desired eigenvectors. For this, taking into account (2.19) and (2.20), we introduce an integer set $\mathbf{R}_0^{(r)}$ for which, with allowance for (2.12), the inequality

$$\max_{s \notin \mathbf{R}_0^{(r)}} |\lambda_r^{(s)}| \geqslant a \gg \varepsilon, \qquad r \leqslant s_0 \tag{3.1}$$

holds. Taking into account (2.7), we represent the desired eigenvectors $\mathbf{X}^{(r)}$ as the sum of two components:

$$\mathbf{X}_1^{(r)} = \sum_{s \in \mathbf{R}_0^{(r)}} \alpha_s^{(r)} \mathbf{X}_0^{(s)}, \qquad \mathbf{X}_2^{(r)} = \sum_{s \notin \mathbf{R}_0^{(r)}} \alpha_s^{(r)} \mathbf{X}_0^{(s)}. \tag{3.2}$$

According to (3.2) we represent (2.15) in the form

$$\sum_{s \in \mathbf{R}_0^{(r)}} (\alpha_s^{(r)})^2 \lambda_r^{(s)} + \sum_{s \notin \mathbf{R}_0^{(r)}} (\alpha_s^{(r)})^2 \lambda_r^{(s)} = O(\varepsilon). \tag{3.3}$$

Assuming that the number of elements S of the set $\mathbf{R}_0^{(r)}$ (the number of similar eigenvalues) is small, so that $S/s_2 = O(1)$, by analogy with (2.16), we find

$$\sum_{s \in \mathbf{R}_0^{(r)}} (\alpha_s^{(r)})^2 \lambda_r^{(s)} = O(\varepsilon). \tag{3.4}$$

We expand the second term (3.3) in two components:

$$\sum_{s \notin \mathbf{R}_0^{(r)}} (\alpha_s^{(r)})^2 \lambda_r^{(s)} = \Pi_1 + \Pi_3$$

(Π_1 is determined in Sec. 2). From (2.16), (2.18), (3.3), and (3.4) it follows that

$$\Pi_1 = O(\varepsilon), \qquad \Pi_3 = O(\varepsilon). \tag{3.5}$$

Taking into account (3.1) and (3.2), from (3.5) we have

$$\|\mathbf{X}_2^{(r)}\|^2 = O(\varepsilon). \tag{3.6}$$

Thus, with allowance for (3.2), the desired eigenvector is representable as

$$\mathbf{X}^{(r)} = \mathbf{X}_1^{(r)} + \sqrt{\varepsilon}\tilde{\mathbf{X}}_2^{(r)}, \qquad \|\mathbf{X}_1^{(r)}\| = O(1), \qquad \|\tilde{\mathbf{X}}_2^{(r)}\| = O(1), \tag{3.7}$$

and the corresponding eigenvalue, according to (2.12) and (2.19), is representable as

$$\lambda^{(r)} = \lambda_0^{(r)}(1 + \tilde{\lambda}^{(r)}), \qquad |\tilde{\lambda}^{(r)}| = O(\varepsilon), \qquad r \leqslant s_0. \tag{3.8}$$

We substitute (3.7) and (3.8) into Eq. (2.1) and multiply it from the left by a rectangular matrix whose rows are vectors $\mathbf{Y}_0^{(p)}$ $[p \in \mathbf{R}_0^{(r)}]$. Taking into account (2.2), (2.3), (2.8), and (2.12), we obtain

$$[\mathrm{diag}\ (\delta\tilde{\lambda}_0^{(r)}) + \mathbf{G} - \tilde{\lambda}^{(r)}\mathbf{J}]\mathbf{X}_1^{(r)} = \eta, \qquad \|\eta\| = O(\varepsilon^{3/2}), \tag{3.9}$$

where $\tilde{\lambda}^{(r)} = (\lambda - \lambda_0^{(r)})/\lambda_0^{(r)}$, $\delta\lambda_0^{(rs)} = (\lambda_0^{(s)} - \lambda_0^{(r)})/\lambda_0^{(r)}$, $\mathbf{J}$ is a unit matrix, and $\mathbf{G}$ is a symmetric real matrix with elements

$$g_{ps}^{(r)} = \frac{\varepsilon_1}{c^{(r)}}\mathbf{Y}_0^{(p)}(\tilde{\mathbf{B}} - \lambda_0^{(r)}\tilde{\mathbf{A}})\mathbf{X}_0^{(s)}, \quad \mathbf{X}_1^{(r)} = \{\alpha_s^{(r)} : s \in \mathbf{R}_0^{(r)}\}, \quad \|\tilde{\mathbf{X}}_1^{(r)}\| = 1 - O(\varepsilon).$$

Using the *a posteriori* estimate of perturbations of eigenvalues for normal matrices [3], from (3.9) we find

$$\tilde{\lambda}^{(r)} = \tilde{\lambda}_1^{(r)} + O(\varepsilon^{3/2}), \qquad |\tilde{\lambda}_1^{(r)}| = \min|\tilde{\lambda}_0^{(t)}| \qquad (t = 1, 2, \ldots, N), \tag{3.10}$$

where $\tilde{\lambda}_0^{(t)}$ are the eigenvalues of the matrix [diag $(\tilde{\lambda}_0^{(r)}) + \mathbf{G}$]. The further solution of the formulated problem is restricted to determining the perturbed component of the eigenvalue $\tilde{\lambda}_1^{(r)}$. However, the solution of the latter problem involves additional difficulties, because, as in [1], the matrices of the aerodynamic influence coefficients $\tilde{\mathbf{A}}_{mn}$ are not determined explicitly. These difficulties are compounded by the fact that, in contrast to the matrices $\tilde{\mathbf{A}}_{mn}^{(0)}$, their perturbed components $\tilde{\mathbf{A}}_{mn}$ do not show the cyclicity property.

4. Determining the Perturbed Components of the Coefficients of the Generalized Hydrodynamic Forces. We consider the quantities

$$\tilde{a}_{ps} = \lambda_0^{(s)} \mathbf{Y}_0^{(p)} \tilde{\mathbf{A}}_1 \mathbf{X}_0^{(s)}, \quad \tilde{\mathbf{A}}_1 = \{\tilde{\mathbf{A}}_{mn}\}_0^{N-1}, \tag{4.1}$$

which enter into the relation for the elements $g_{ps}^{(r)}$ of the matrix $\mathbf{G}$ of Eq. (3.9). The value of $\tilde{a}_{ps}$ determines the perturbed component of the coefficient of the generalized hydrodynamic force that corresponds to the pth generalized coordinate and occurs with cascade vibrations of the sth form.

Assuming that the vectors $\mathbf{X}_0^{(s)}$ are known [1], we shall seek the vectors

$$\tilde{\mathbf{P}}^{(s)} = \lambda_0^{(s)} \tilde{\mathbf{A}}_1 \mathbf{X}_0^{(s)}, \tag{4.2}$$

which determine the nonstationary components of the distributed hydrodynamic load that acts on the cascade blades during cascade vibrations with a frequency $\omega_0^{(s)} = \sqrt{\lambda_0^{(s)}}$ and by the form $\mathbf{X}_0^{(s)}$, which correspond to natural vibrations of a uniform cascade. As is known [5], natural vibrations of the blades of a uniform cascade proceed with the same amplitude and a constant phase shift $\mu_k = 2\pi k/N$ $(k = 0, 1, \ldots, N-1)$ between vibrations of neighboring blades. The indicated forms of cascade vibrations are due to the hydrodynamic interaction of the blades, and natural vibrations of the cascade with N different phase shifts occur in the vicinity of each mode of natural vibrations of an isolated blade. Bearing this in mind, we expand the full spectrum of eigenvalues of system (2.1) into an aggregate of subsets, $\Lambda^{(j)} = \{\lambda_0^{(jk)}\}_{k=0}^{N-1}$, whose element indices correspond to the initial indices as follows:

$$r = 1 + k + (j-1)N, \qquad j = 1, 2, \ldots, N_0, \tag{4.3}$$

where j denotes the mode number. By analogy with (4.3), we put the indices p and s in correspondence with the indices (uq) and (vl), so that

$$p = 1 + q + (u-1)N, \quad s = 1 + l + (v-1)N \tag{4.4}$$
$$(q, l = 0, 1, \ldots, N-1; \quad u, v = 1, 2, \ldots, N_0).$$

According to the indicated law of vibrations, we represent the vectors $\mathbf{X}_0^{(s)}$ and $\tilde{\mathbf{P}}^{(s)}$ as follows:

$$\mathbf{X}_0^{(s)} = \mathbf{X}_0^{(vl)} = \frac{1}{\sqrt{N}} \{\mathbf{X}_{0n}^{(vl)}\}_{n=0}^{N-1}, \qquad \tilde{\mathbf{P}}^{(s)} = \tilde{\mathbf{P}}^{(vl)} = \frac{1}{\sqrt{N}} \{\tilde{\mathbf{P}}_n^{(vl)}\}_{n=0}^{N-1}; \tag{4.5}$$

$$\mathbf{X}_{0n}^{(vl)} = \mathbf{X}_{00}^{(vl)} e^{i\mu_l n}, \qquad \|\mathbf{X}_{(0n)}^{(vl)}\| = 1; \tag{4.6}$$

$$\tilde{\mathbf{P}}_m^{(vl)} = \lambda_0^{(vl)} \sum_{n=0}^{N-1} \tilde{\mathbf{A}}_{mn} \mathbf{X}_{00}^{(vl)} e^{i\mu_l n}. \tag{4.7}$$

The subvector $\tilde{\mathbf{P}}_m^{(vl)}$ determines the perturbed component of the distributed unsteady hydrodynamic load that acts on the mth blade when the cascade vibrations obey the above law. Thus, the determination of the coefficients $\tilde{a}_{ps}$ (4.1) reduces to solving the hydrodynamic problem of unsteady flow past the blade cascade vibrating according to the specified law, and does not require the finding of the matrices in explicit form.

5. Determining the Perturbed Components of the Eigenvalues. According to (3.9) and (3.10), the first-order perturbations of the eigenvalues can be determined by solving the eigenvalue problem for the matrix $\mathbf{\Delta} + \mathbf{G}$, where

$$\mathbf{\Delta} = \text{diag}\,(\lambda_0^{(rs)}), \qquad \mathbf{G} = \{g_{ps}^{(r)}\}, \qquad p, s \in \mathbf{R}_0^{(r)}. \tag{5.1}$$

Taking into account (4.3)–(4.7), it is convenient to represent the elements of the matrix $\mathbf{G}$ in the new indices:

$$g_{uv}^{(ql)} = \frac{\varepsilon_1}{c_j^{(k)}}(h_{uv}^{(ql)} - a_{uv}^{(ql)}). \tag{5.2}$$

Here

$$\begin{gathered} a_{uv}^{(ql)} = \sum_{m=0}^{N-1} \mathbf{Y}_{00}^{(uq)} \tilde{\mathbf{P}}_m^{(vl)} e^{-i\mu_q m}, \\ c_j^{(k)} = \mathbf{Y}_{00}^{(jk)} \mathbf{C} \mathbf{X}_{00}^{(jk)}, \quad h_{uv}^{(ql)} = \sum_{m=0}^{N-1} \mathbf{Y}_{00}^{(uq)} (\tilde{\mathbf{C}}_m - \lambda_0^{(jk)} \tilde{\mathbf{M}}_m) \mathbf{X}_{00}^{(vl)} e^{i\mu_m(l-q)}. \end{gathered} \tag{5.3}$$

We consider some particular cases of similarity between the eigenvalues of a uniform cascade, which can be used as a basis for the algorithm of determining eigenvalues in the general case.

1. **Isolated Eigenvalue.** The perturbation $\tilde{\lambda}_1^{(jk)}$ of the eigenvalue $\lambda_0^{(jk)}$, for which the set $\mathbf{R}_0^{(jk)}$ contains one element, according to (3.1), (3.9), and (5.1)–(5.3), has the form $\tilde{\lambda}_1^{(jk)} = g_{jj}^{(kk)}$.

2. **Similarity between Two Eigenvalues under Strong Hydrodynamic Interaction of Blades.** Under strong hydrodynamic interaction for the fixed jth vibration mode of a uniform cascade, the eigenvalues that correspond to blade vibrations with different phase shifts will be sufficiently isolated from one another. In this case, let several eigenvalues for different vibration modes be close to one another so that

$$\lambda_0^{(vk)} = \lambda_0^{(jk)}(1 + \varepsilon\lambda_v'), \qquad |\lambda_v'| = O(1), \quad v \in \mathbf{R}_0^{(jk)},$$

where $\mathbf{R}_0^{(jk)}$ is the set of values of v that correspond to the index of similar eigenvalues $\lambda_0^{(vk)}$. According to (5.1)–(5.3), the first-order perturbations of these eigenvalues coincide with the eigenvalues of the matrix $\Delta + \mathbf{G}$, where

$$\Delta = \varepsilon_1 \mathrm{diag}\, \{\lambda_v'\}_{v\in\mathbf{R}_0^{(jk)}}, \qquad \mathbf{G} = \{g_{uv}^{(kk)}\}_{u,v\in\mathbf{R}_0^{(jk)}}. \tag{5.4}$$

3. **Weak Hydrodynamic Interaction for Isolated Eigenvalues of Blade Vibrations in Vacuum.** Let the hydrodynamic interaction of the blades be small so that

$$\|\mathbf{A}_{mn}\| = \varepsilon_1 O(\|\mathbf{M}\|), \qquad m \neq n. \tag{5.5}$$

This is the case for hydraulic-turbine cascades of sufficiently large pitch-chord ratio and for almost all gas-turbomachine cascades. In addition, let the natural frequencies of blade vibrations in vacuum be sufficiently isolated, so that

$$|\lambda_0^{(lk)} - \lambda_0^{(jk)}|/\lambda_0^{(jk)} \geqslant a \gg \varepsilon, \qquad l \neq j. \tag{5.6}$$

Taking into account (1.5)–(1.7), from (5.5) we obtain the estimate

$$\|\mathbf{A}_{mn}^{(0)}\| = \varepsilon_1 O(\|\mathbf{M}\|), \qquad \|\tilde{\mathbf{A}}_{mn}\| = \varepsilon_1 O(\|\mathbf{M}\|), \qquad m \neq n. \tag{5.7}$$

With allowance for (5.7), the blocks $\mathbf{A}_{mn}^{(0)}$ for $m \neq n$ of the matrix $\mathbf{A}$ in Eq. (2.1) can be transposed to the right side, and the blocks $\tilde{\mathbf{A}}_{mn}$ $(m \neq n)$ can be ignored. Then, taking into account (5.6), we can conclude that the set $\mathbf{R}_0^{(j0)}$, according to (4.3), consists of the numbers $s = 1 + l + (j-1)N$, $l = 0, 1, \ldots, N-1$. This means that, for a certain jth mode of blade vibrations, the set of eigenvalues due to the hydrodynamic interaction of the blades, with allowance for their nonuniformity, can be considered as perturbed values of $\lambda_0^{(j0)}$. Their perturbed components $\tilde{\lambda}_1^{(j0)}$ are determined by solving the eigenvalue problem for the matrix $\mathbf{G}^{(j)} = \{g_{jj}^{(ql)}\}_{q,l=0}^{N-1}$ (in this case, $\Delta \equiv 0$). Taking into account the above circumstance, with allowance for (1.6) and (4.2)–(4.7), the quantities (5.3), which determine the elements of the matrix $\mathbf{G}$, are represented as

$$a_{jj}^{(ql)} = \mathbf{Y}_{00}^{(j0)} \sum_{m=0}^{N-1} \Big[\sum_{k=0}^{N-1} \left(\mathbf{P}_{00}^{(jl)} - \mathbf{P}_{00}^{(jk)} \right) + \tilde{\mathbf{P}}_m^{(j0)} \Big] e^{i\mu_m(l-q)}, \tag{5.8}$$

$$h_{jj}^{(ql)} = \mathbf{Y}_{00}^{(j0)}(\tilde{\mathbf{C}}_m - \lambda_0^{(j0)}\tilde{\mathbf{M}}_m)\mathbf{X}_{00}^{(j0)} e^{i\mu_m(l-q)},$$

where $\mathbf{P}_{00}^{(jk)}$ is a vector that gives the amplitude function of the unsteady hydrodynamic force that acts on the original blade of a uniform cascade vibrating according to the jth mode with a phase shift μ_k, and $\tilde{\mathbf{P}}_m^{(j0)}$ is the perturbed component of this vector that is due to the nonuniformity of the cascade for the mth blade, the jth mode, and zero phase shift.

The matrix equation from which the desired values of $\tilde{\lambda}_1^{(j0)}$ are found is conveniently rearranged as

$$(\mathbf{J}\tilde{\lambda}_1^{(j0)} - \mathbf{E}^{-1}\mathbf{G}^{(j)}\mathbf{E})\mathbf{Z}^{(j)} = 0. \tag{5.9}$$

Here

$$\mathbf{E} = \frac{1}{\sqrt{N}}\left\{\exp\left(i\frac{2\pi}{N}mn\right)\right\}_{m,n=0}^{N-1}, \quad \mathbf{Z}^{(j)} = \mathbf{E}^{-1}\mathbf{X}_1^{(j)} = \{z_n^{(j)}\}_{n=0}^{N-1},$$

and $\mathbf{Z}^{(j)}$ is an eigenvector of (5.9). It corresponds to one of the eigenvalues of the matrix $\mathbf{G}^{(j)}$ and the nth component $z_n^{(j)}$ determines the vibration amplitude of the nth blade for $\|\mathbf{Z}^{(j)}\| = 1$. Performing a similarity transformation of the matrix $\mathbf{G}$ and taking into account (5.8), we have

$$\mathbf{E}^{-1}\mathbf{G}^{(j)}\mathbf{E} = \mathbf{L}^{(j)} + \operatorname{diag}(\tilde{r}_m^{(j)}), \tag{5.10}$$

where $\mathbf{L}^{(j)}$ is a cyclic matrix with the elements

$$l_{mn}^{(j)} = \frac{\varepsilon_1}{c_j^{(0)}}\mathbf{Y}_{00}^{(j0)}\sum_{k=0}^{N-1}\mathbf{P}_{00}^{(jk)}(1 - e^{-i\mu_k(n-m)}); \tag{5.11}$$

$$\tilde{r}_m^{(j)} = \frac{\varepsilon_1}{c_j^{(0)}}\mathbf{Y}_{00}^{(j0)}[(\tilde{\mathbf{C}}_m - \lambda_0^{(j0)}\tilde{\mathbf{M}}_m)\mathbf{X}_{00}^{(j0)} - \tilde{\mathbf{P}}_m^{(j0)}]. \tag{5.12}$$

Note that equations that are similar in form to (5.9)–(5.12) were obtained in studies of the stability of vibrations of nonuniform cascades in a gas flow [5–7]. However, the mechanical model of the hydroelastic vibrations considered differs from the model of cascade vibrations in a gas flow. This difference is as follows:

(1) the frequency and form of blade vibrations depend significantly on the interaction of the blades with the liquid,

(2) the mistuning parameter of the natural frequencies of blade vibrations $\tilde{r}_m$ can depend on the perturbation of hydrodynamic forces due to the geometric nonuniformity of the cascade.

Because of the above-mentioned circumstances, for a hydraulic-turbine cascade, the problem of elastic vibrations of blades with allowance for the hydrodynamic forces acting on the blades should be solved jointly with the problem of unsteady flow through the cascade.

4. General Case of Similarity between Eigenvalues. Let, in the hydrodynamic interaction of blades, several eigenvalues of blade vibrations in vacuum be close to one another. This case is fairly typical for cascade vibrations in a gas flow, which were considered in [8].

With allowance for (3.9), (5.4), (5.9), and (5.10), the first-order perturbations of the set of eigenvalues studied will coincide with the eigenvalues of the block matrix $[\mathbf{L} + \operatorname{diag}(\mathbf{\Delta} + \tilde{\mathbf{R}}_m)]$, where $\mathbf{L} = \{\mathbf{L}_{mn}\}_{m,n=0}^{N-1}$, and $\mathbf{L}_{mn}$ are square matrices of the order of the number of similar eigenvalues for the jth mode, with the elements

$$l_{mn}^{(uv)} = \frac{\varepsilon_1}{c_v^{(0)}}\mathbf{Y}_{00}^{(u0)}\sum_{k=0}^{N-1}\mathbf{P}_{00}^{(vk)}(1 - e^{-i\mu_k(n-m)}),$$

and $\tilde{\mathbf{R}}_m$ are square matrices of the same order as $\mathbf{L}_{mn}$, with the elements

$$\tilde{r}_{mn}^{(uv)} = \frac{\varepsilon_1}{c_v^{(0)}}\mathbf{Y}_{00}^{(u0)}[(\tilde{\mathbf{C}}_m - \tilde{\lambda}_0^{(v0)}\tilde{\mathbf{M}}_m)\mathbf{X}_{00}^{(v0)} - \tilde{\mathbf{P}}_m^{(v0)}].$$

TABLE 1

Tone number	f, Hz				
	Perturbation method				Iteration method
	$\mu = 0$	$\mu_1 = \pi/3$	$\mu_2 = 2\pi/3$	$\mu_3 = \pi$	$\mu_3 = \pi$
1	471	458.6	453.3	446.3	448
2	714	707	700	698	700
3	1124	1123	1122.3	1121.8	1122

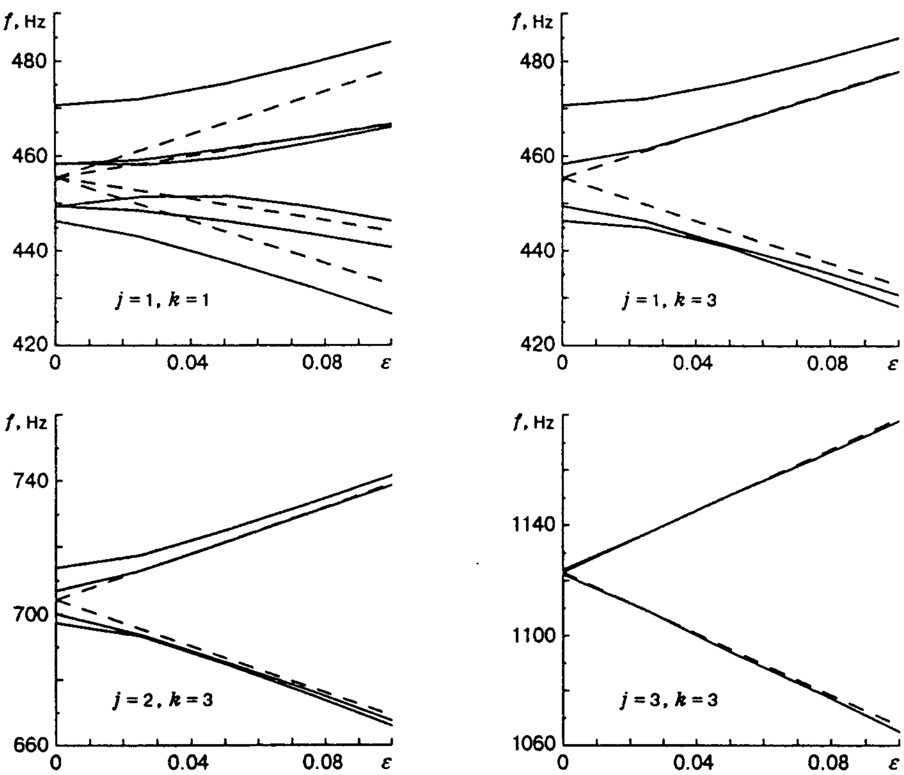

Fig. 1

6. Effect of Small Geometric Nonuniformity of a Cascade on the Amplitude–Frequency Characteristics of Cascade Vibrations. The results of solution of the eigenvalue problem considered are of practical interest mainly for tuning away from resonance phenomena, which can occur with some sources of excitation. As is known the level of resonance vibrations depends not only on the damping in the system but also on the form of exciting force. That is, in the resonance regime with specified damping, the vibration amplitude of the system is proportional to the scalar product of the distributed exciting force function by the form function for the corresponding natural vibrations. For example, for turbomachine cascades, the main source of excitation is the unsteady hydrodynamic forces due to the peripheral nonuniformity of the flow. These forces are a set of harmonics each of which acts on all neighboring blades at equal frequencies and amplitudes and also with a constant phase shit μ_k.

All natural vibration forms of a uniform cascade show generalized periodicity. Consequently, for resonance of forced vibrations of the cascade under the action of the indicated exciting forces, it is necessary that, in addition to coincidence of the frequencies of the exciting forces with the natural frequencies, the corresponding phase shifts of the forces also coincide. In other words, for each individual harmonic of the exciting force, only one eigenvalue from the set of eigenvalues $\Lambda^{(j)}$, due to the hydrodynamic interaction of blades, will be resonant.

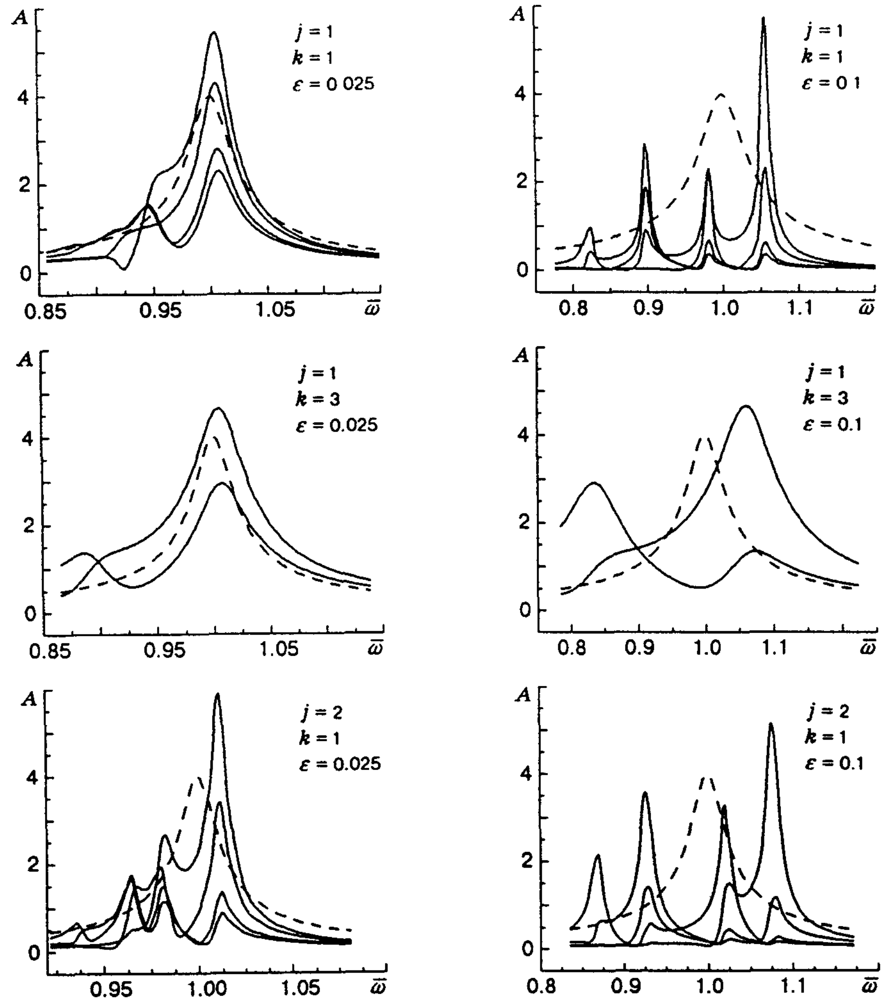

Fig. 2

The character of forced vibrations of nonuniform cascades is fundamentally different. From the above algorithms of solution of the eigenvalue problem it follows that the vibration forms of nonuniform cascades generally do not show generalized periodicity. Therefore, for each harmonic of the exciting force, resonance might be expected when its frequency coincides with any natural frequency of cascade vibrations. In this case, the vibration amplitudes of different blades differ from one another. The expected regularities can be readily illustrated by amplitude–frequency characteristics of forced vibrations of cascades.

As an example, we studied numerically the effect of cascade nonuniformity on free and forced blade vibrations for a PL 587V hydraulic-turbine model, considered in [1]. The cascade consists of six blades. The calculation was performed for a blade angle that corresponds to the operating position of the cascade.

Table 1 gives the results of calculating the dependence of the natural vibration frequency of the uniform cascade on the phase shift μ_k using the perturbation method, and a comparison with the results of calculation using the iteration method [1].

According to the results of Table 1, calculations for vibrations of the nonuniform cascade were performed by the algorithm (5.9)–(5.12). In the indicated algorithm, the nonuniformity parameters of the cascade are the elements $\bar{r}_m^{(j)}$ of the matrix (5.10), which characterize the mistuning of the natural frequencies of the cascade vibrations. For the given geometric nonuniformity of the cascade, they are determined from formula (5.12). They can also be determined approximately from experimental measurements of the natural frequencies of isolated blades $\omega_m^{(j)}$ (in the absence of their hydrodynamic interaction). In this case, it follows from (5.9) that

$$\tilde{r}_m^{(j)} = 2\Delta\omega_m^{(j)}/\omega^{(j)}, \quad \Delta\omega^{(j)} = \omega_m^{(j)} - \omega^{(j)}, \quad \omega^{(j)} = \sum_{m=0}^{N-1} \omega_m^{(j)}/N.$$

Examples of dependences of the natural frequencies of this cascade f on its nonuniformity parameters are shown in Fig. 1 by solid curves. The dashed curves show dependences of the vibration frequencies of individual blades (ignoring their interaction with one another), which govern the corresponding frequency mistuning of the cascade. In this case, the results correspond to the mistuning law of the form

$$\tilde{r}_m^{(j)} = \varepsilon \cos \frac{2\pi}{N} mk \qquad (N = 6). \tag{6.1}$$

It can be seen from the above dependences that with increase in mistuning and the mode number, the natural frequencies of the cascade become similar to the "partial" frequencies.

Figure 2 shows amplitude–frequency characteristics (solid curves) for various blades of nonuniform cascades for the case where the exciting forces act on the blade with the same intensity and zero phase shit between the action on different blades (this case is typical of stand tests of blade cascades of hydraulic turbines). Artificial damping of blade vibrations is introduced into the calculated model to restrict the maximum values of vibration amplitudes. The dashed curves are the amplitude–frequency characteristics of the corresponding uniform cascade. The mistuning parameters given in Fig. 2 correspond to the law (6.1). It is evident that the maximum vibration amplitudes of individual blades can far exceed the corresponding vibration amplitudes of uniform cascades. In addition, as noted above, for nonuniform cascades, in contrast to uniform cascades (dashed curves), the resonance phenomenon occurs for all natural frequencies of cascade vibrations (the so-called separating effect of natural frequencies).

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